History-Preserving Bisimilarity for Higher-Dimensional Automata via Open Maps

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Abstract

We show that history-preserving bisimilarity for higher-dimensional automata has a simple characterization directly in terms of higher-dimensional transitions. This implies that it is decidable for finite higher-dimensional automata. To arrive at our characterization, we apply the open-maps framework of Joyal, Nielsen and Winskel in the category of unfoldings of precubical sets.

Keywords: higher-dimensional automaton, history-preserving bisimilarity, homotopy, unfolding, concurrency

1 Introduction

The dominant notion for behavioral equivalence of processes is *bisimulation* as introduced by Park [30] and Milner [26]. It is compelling because it enjoys good algebraic properties, admits several easy characterizations using modal logics, fixed points, or game theory, and generally has low computational complexity.

Bisimulation, or rather its underlying semantic model of *transition systems*, applies to a setting in which concurrency of actions is the same as non-deterministic interleaving; using CCS notation [26], $a|b = a.b+b.a$. For some applications however, a distinction between these two is necessary, which has led to development of so-called *non-interleaving* or *truly concurrent* models such as Petri nets [31], event structures [29], asynchronous transition systems [4, 34] and others; see [39] for a survey.

One of the most popular notions of equivalence for non-interleaving systems is *history-preserving bisimilarity* (or *hp-bisimilarity* for short). It was introduced independently by Degano, De Nicola and Montanari in [6] and by Rabinovich and Trakhtenbrot [33] and then for event structures by van Glabbeek and Goltz in [38] and for Petri nets by Best *et.al.* in [5]. One reason for its popularity is that it is a congruence under action refinement [5,38], another its good decidability properties: it has been shown to be decidable for safe Petri nets by Montanari and Pistore [28]. As a contrast, its cousin *hereditary* hp-bisimilarity is shown undecidable for 1-safe Petri nets by Jurdziński, Nielsen and Srba in [23].

Higher-dimensional automata (or *HDA*) is another non-interleaving formalism for reasoning about behavior of concurrent systems. Introduced by Pratt [32] and van Glabbeek [36] in 1991 for the purpose of a *geometric* interpretation to the theory of concurrency, it has since been shown by van Glabbeek [37] that HDA provide a generalization (up to hp-bisimilarity) to "the main models of concurrency proposed in the literature" [37], including the ones mentioned above. Hence HDA are useful as a tool for comparing and relating different models, and also as a modeling formalism by themselves.

HDA are geometric in the sense that they are very similar to the *simplicial complexes* used in algebraic topology, and research on HDA has drawn on a lot of tools and methods from geometry and algebraic topology such as homotopy [10,13], homology [14,19], and model categories [15,16], see also the survey [17].

In this paper we give a geometric interpretation to hp-bisimilarity for HDA, using the open-maps approach introduced by Joyal, Nielsen and Winskel in [22] and results from a previous paper [7] by the first author. Using this interpretation, we show that hp-bisimilarity for HDA has a characterization directly in terms of (higher-dimensional) *transitions* of the HDA, rather than in terms of runs as *e.g.* for Petri nets [12].

Our results imply *decidability* of hp-bisimilarity for finite HDA. They also put hp-bisimilarity firmly into the open-maps framework of [22] and tighten the connections between bisimilarity and weak topological *fibrations* [3,24].

Due to lack of space, we have had to confer all proofs of this paper to a separate appendix.

2 Higher-Dimensional Automata

As a formalism for concurrent behavior, HDA have the specific feature that they can express all higher-order dependencies between events in a concurrent system. Like for transition systems, they consist of states and transitions which are labeled with events. Now if two transitions from a state, with labels a and b for example, are independent, then this is expressed by the existence of a *two-dimensional* transition with label ab. Fig. 1 shows two examples; on the left, transitions a and b are independent, on the right, they can merely be executed in any order. Hence for HDA, as indeed for any formalism employing the so-called *true concurrency* paradigm, the algebraic law $a|b = a.b + b.a$ does *not* hold; concurrency is not the same as interleaving.

The above considerations can equally be applied to sets of more than two events: if three events a, b, c are independent, then this is expressed using a three-dimensional transition labeled abc. Hence this is different from mutual pairwise independence (expressed by transitions ab, ac, bc), a distinction which cannot be

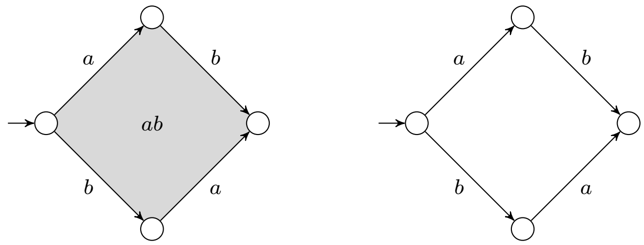

Fig. 1. HDA for the CCS expressions $a|b$ (left) and $a.b + b.a$ (right). In the left HDA, the square is filled in by a two-dimensional transition labeled ab, signifying independence of events a and b. On the right, a and b are not independent.

made in formalisms such as asynchronous transition systems [4, 34] or transition systems with independence [39] which only consider binary independence relations.

An unlabeled HDA is essentially a pointed precubical set as defined below. For labeled HDA, one can pass to an arrow category; this is what we shall do in Section 6. Until then, we concentrate on the unlabeled case.

A *precubical set* is a graded set $X = \{X_n\}_{n\in\mathbb{N}}$ together with mappings $\delta_k^\nu : X_n \to X_{n-1}$, $k \in \{1, \dots, n\}$, $\nu \in \{0, 1\}$, satisfying the *precubical identity*

$$\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu \qquad (k < \ell)\,. \tag{1}$$

The mappings δ_k^ν are called *face maps*, and elements of X_n are called *n-cubes*. As above, we shall usually omit the extra subscript (n) in the face maps. Faces $\delta_k^0 x$ of an element $x \in X$ are to be thought of as *lower faces*, $\delta_k^1 x$ as *upper faces*. The precubical identity expresses the fact that $(n-1)$-faces of an n-cube meet in common $(n-2)$-faces, see Fig. 2 for an example of a 2-cube and its faces.

Morphisms $f : X \to Y$ of precubical sets are graded mappings $f = \{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ which commute with the face maps: $\delta_k^\nu \circ f_n = f_{n-1} \circ \delta_k^\nu$ for all $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$, $\nu \in \{0, 1\}$. This defines a category pCub of precubical sets and morphisms.

A *pointed* precubical set is a precubical set X with a specified 0-cube $i \in X_0$, and a pointed morphism is one which respects the point. This defines a category which is isomorphic to the comma category $* \downarrow \mathsf{pCub}$, where $* \in \mathsf{pCub}$ is the precubical set with one 0-cube and no other n-cubes. Note that $*$ is *not* terminal in pCub (instead, the terminal object is the infinite-dimensional precubical set with one cube in every dimension).

Definition 2.1 The category of *higher-dimensional automata* is the comma cate-

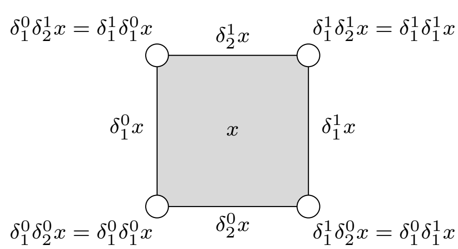

Fig. 2. A 2-cube x with its four faces $\delta_1^0 x$, $\delta_1^1 x$, $\delta_2^0 x$, $\delta_2^1 x$ and four corners.

gory $\mathsf{HDA} = * \downarrow \mathsf{pCub}$, with objects pointed precubical sets and morphisms commutative diagrams

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}.$$

Hence a one-dimensional HDA is a transition system; indeed, the category of transition systems [39] is isomorphic to the full subcategory of HDA spanned by the one-dimensional objects. Similarly one can show [18] that the category of asynchronous transition systems is isomorphic to the full subcategory of HDA spanned by the (at most) two-dimensional objects. The category HDA as defined above was used in [7] to provide a categorical framework (in the spirit of [39]) for parallel composition of HDA. In this article we also introduced a notion of bisimilarity which we will review in the next section.

3 Path Objects, Open Maps and Bisimilarity

With the purpose of introducing bisimilarity via *open maps* in the sense of [22], we identify here a subcategory of HDA consisting of path objects and path-extending morphisms. We say that a precubical set X is a *precubical path object* if there is a (necessarily unique) sequence $(x_1, \dots, x_m)$ of elements in X such that $x_i \neq x_j$ for $i \neq j$,

- for each $x \in X$ there is $j \in \{1, \dots, m\}$ for which $x = \delta_{k_1}^{\nu_1} \cdots \delta_{k_p}^{\nu_p} x_j$ for some indices $\nu_1, \dots, \nu_p$ and a *unique* sequence $k_1 < \cdots < k_p$, and
- for each $j = 1, \dots, m-1$, there is $k \in \mathbb{N}$ for which $x_j = \delta_k^0 x_{j+1}$ or $x_{j+1} = \delta_k^1 x_j$.

Note that precubical path objects are *non-selflinked* in the sense of [10]. If X and Y are precubical path objects with representations $(x_1, \dots, x_m)$, $(y_1, \dots, y_p)$, then a morphism $f : X \to Y$ is called a *cube path extension* if $x_j = y_j$ for all $j = 1, \dots, m$ (hence $m \leq p$).

Definition 3.1 The category HDP of *higher-dimensional paths* is the subcategory of HDA which as objects has pointed precubical paths, and whose morphisms are generated by isomorphisms and pointed cube path extensions.

A *cube path* in a precubical set X is a morphism $P \to X$ from a precubical path object P. In elementary terms, this is a sequence $(x_1, \dots, x_m)$ of elements of X such that for each $j = 1, \dots, m-1$, there is $k \in \mathbb{N}$ for which $x_j = \delta_k^0 x_{j+1}$ (start of new part of a computation) or $x_{j+1} = \delta_k^1 x_j$ (end of a computation part). We show an example of a cube path in Fig. 3.

A cube path in a HDA $i : * \to X$ is *pointed* if $x_1 = i$, hence if it is a pointed morphism $P \to X$ from a higher-dimensional path P. We will say that a cube path $(x_1, \dots, x_m)$ is *from* x_1 *to* x_m, and that a cube $x \in X$ in a HDA X is *reachable* if there is a pointed cube path to x in X.

Cube paths can be *concatenated* if the end of one is compatible with the beginning of the other: If $\rho = (x_1, \dots, x_m)$ and $\sigma = (y_1, \dots, y_p)$ are cube paths with

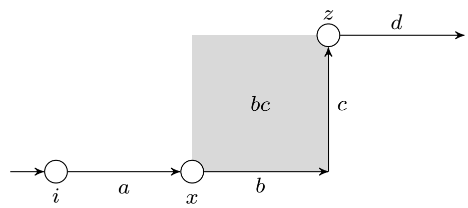

Fig. 3. Graphical representation of the two-dimensional cube path $(i, a, x, b, bc, c, z, d)$. Its computational interpretation is that a is executed first, then execution of b starts, and while b is running, c starts to execute. After this, b finishes, then c, and then execution of d is started. Note that the computation is partial, as d does not finish.

$y_1 = \delta_k^1 x_m$ or $x_m = \delta_k^0 y_1$ for some k, then their *concatenation* is the cube path $\rho * \sigma = (x_1, \dots, x_m, y_1, \dots, y_p)$. We say that ρ is a *prefix* of χ and write $\rho \sqsubseteq \chi$ if there is a cube path ρ for which $\chi = \rho * \sigma$.

Definition 3.2 A pointed morphism $f : X \to Y$ in HDA is an *open map* if it has the right lifting property with respect to HDP, *i.e.* if it is the case that there is a lift r in any commutative diagram as below, for morphisms $g : P \to Q \in \mathsf{HDP}$, $p : P \to X, q : Q \to Y \in \mathsf{HDA}$:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] \arrow[ur, dotted, "r"] & Y \end{tikzcd}$$

HDA X, Y are *bisimilar* if there is $Z \in \mathsf{HDA}$ and a span of open maps $X \leftarrow Z \to Y$ in HDA.

It follows straight from the definition that composites of open maps are again open. By the next lemma, morphisms are open precisely when they have a zig-zag property similar to the one of [22].

Lemma 3.3 *For a morphism $f : X \to Y \in \mathsf{HDA}$, the following are equivalent:*

(i) *f is open;*

(ii) *for any reachable $x_1 \in X$ and any $y_2 \in Y$ with $f(x_1) = \delta_k^0 y_2$ for some k, there is $x_2 \in X$ for which $x_1 = \delta_k^0 x_2$ and $y_2 = f(x_2)$;*

(iii) *for any reachable $x_1 \in X$ and any cube path $(y_1, \dots, y_m)$ in Y with $y_1 = f(x_1)$, there is a cube path $(x_1, \dots, x_m)$ in X for which $y_j = f(x_j)$ for all $j = 1, \dots, m$.*

Theorem 3.4 *For HDA $i : * \to X$, $j : * \to Y$, the following are equivalent:*

(i) *X and Y are bisimilar;*

(ii) *there exists a precubical subset $R \subseteq X \times Y$ for which $(i, j) \in R$, and such that for all reachable $x_1 \in X$, $y_1 \in Y$ with $(x_1, y_1) \in R$,*
   - *for any $x_2 \in X$ for which $x_1 = \delta_k^0 x_2$ for some k, there exists $y_2 \in Y$ for which $y_1 = \delta_k^0 y_2$ and $(x_2, y_2) \in R$,*
   - *for any $y_2 \in Y$ for which $y_1 = \delta_k^0 y_2$ for some k, there exists $x_2 \in X$ for which $x_1 = \delta_k^0 x_2$ and $(x_2, y_2) \in R$;*

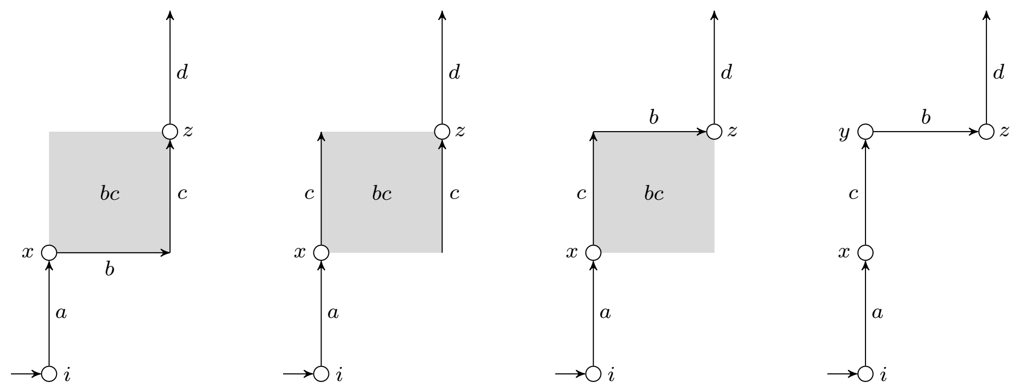

Fig. 4. Graphical representation of the cube path homotopy $(i, a, x, b, bc, c, z, d) \sim (i, a, x, c, bc, c, z, d) \sim (i, a, x, c, bc, b, z, d) \sim (i, a, x, c, y, b, z, d)$.

(iii) *there exists a precubical subset $R \subseteq X \times Y$ for which $(i, j) \in R$, and such that for all reachable $x_1 \in X$, $y_1 \in Y$ with $(x_1, y_1) \in R$,*
- *for any cube path $(x_1, \dots, x_m)$ in X, there exists a cube path $(y_1, \dots, y_m)$ in Y with $(x_p, y_p) \in R$ for all $p = 1, \dots, m$,*
- *for any cube path $(y_1, \dots, y_m)$ in Y, there exists a cube path $(x_1, \dots, x_m)$ in X with $(x_p, y_p) \in R$ for all $p = 1, \dots, m$.*

Note that the requirement that R be a precubical subset, in items (ii) and (iii) above, is equivalent to saying that whenever $(x, y) \in R$, then also $(\delta^\nu_k x, \delta^\nu_k y) \in R$ for any k and $\nu \in \{0, 1\}$.

4 Homotopies and Unfoldings

In order to reason about hp-bisimilarity, we need to introduce in which cases different cube paths are equivalent due to independence of actions. Following [37], we model this equivalence by a combinatorial version of *homotopy* which is an extension of the equivalence defining *Mazurkiewicz traces* [25].

We say that cube paths $(x_1, \dots, x_m)$, $(y_1, \dots, y_m)$ are *adjacent* if $x_1 = y_1$, $x_m = y_m$, there is precisely one index $p \in \{1, \dots, m\}$ at which $x_p \neq y_p$, and

- $x_{p-1} = \delta^0_k x_p$, $x_p = \delta^0_\ell x_{p+1}$, $y_{p-1} = \delta^0_{\ell-1} y_p$, and $y_p = \delta^0_k y_{p+1}$ for some $k < \ell$, or vice versa,
- $x_p = \delta^1_k x_{p-1}$, $x_{p+1} = \delta^1_\ell x_p$, $y_p = \delta^1_{\ell-1} y_{p-1}$, and $y_{p+1} = \delta^1_k y_p$ for some $k < \ell$, or vice versa,
- $x_p = \delta^0_k \delta^1_\ell y_p$, $y_{p-1} = \delta^0_k y_p$, and $y_{p+1} = \delta^1_\ell y_p$ for some $k < \ell$, or vice versa, or
- $x_p = \delta^1_k \delta^0_\ell y_p$, $y_{p-1} = \delta^0_\ell y_p$, and $y_{p+1} = \delta^1_k y_p$ for some $k < \ell$, or vice versa.

Homotopy of cube paths is the reflexive, transitive closure of the adjacency relation. We denote homotopy of cube paths using the symbol $\sim$, and the homotopy class of a cube path $(x_1, \dots, x_m)$ is denoted $[x_1, \dots, x_m]$. The intuition of adjacency is rather simple, even though the combinatorics may look complicated, see Fig. 4. Note that adjacencies come in two basic “flavors”: the first two above in which the dimensions of x_ℓ and y_ℓ are the same, and the last two in which they differ by 2.

The following lemma shows that, as expected, cube paths entirely contained in one cube are homotopic (provided that they share endpoints).

Lemma 4.1 *Let $x \in X_n$ in a precubical set X and $(k_1, \dots, k_n)$, $(\ell_1, \dots, \ell_n)$ sequences of indices with $k_j, \ell_j \le j$ for all $j = 1, \dots, n$. Let $x_j = \delta^0_{k_j} \cdots \delta^0_{k_n} x$, $y_j = \delta^0_{\ell_j} \cdots \delta^0_{\ell_n} x$. Then the cube paths $(x_1, \dots, x_n, x) \sim (y_1, \dots, y_n, x)$.*

We extend concatenation and prefix to homotopy classes of cube paths by defining $[x_1, \dots, x_m] * [y_1, \dots, y_p] = [x_1, \dots, x_m, y_1, \dots, y_p]$ and saying that $\tilde{x} \sqsubseteq \tilde{z}$, for homotopy classes $\tilde{x}$, $\tilde{z}$ of cube paths, if there are $(x_1, \dots, x_m) \in \tilde{x}$ and $(z_1, \dots, z_q) \in \tilde{z}$ for which $(x_1, \dots, x_m) \sqsubseteq (z_1, \dots, z_q)$. It is easy to see that concatenation is well-defined, and that $\tilde{x} \sqsubseteq \tilde{z}$ if and only if there is a homotopy class $\tilde{y}$ for which $\tilde{z} = \tilde{x} * \tilde{y}$.

Using homotopy classes of cube paths, we can now define the *unfolding* of a HDA. Unfoldings of HDA are similar to unfoldings of transition systems [39] or Petri nets [21, 29], but also to *universal covering spaces* in algebraic topology. The intention is that the unfolding of a HDA captures all its computations, up to homotopy.

We say that a HDA X is a *higher-dimensional tree* if it holds that for any $x \in X$, there is precisely one homotopy class of pointed cube paths to x. The full subcategory of HDA spanned by the higher-dimensional trees is denoted HDT. Note that any higher-dimensional path is a higher-dimensional tree; indeed there is an inclusion $\mathsf{HDP} \hookrightarrow \mathsf{HDT}$.

Definition 4.2 The *unfolding* of a HDA $i : * \to X$ consists of a HDA $\tilde{i} : * \to \tilde{X}$ and a pointed *projection* morphism $\pi_X : \tilde{X} \to X$, which are defined as follows:

- $\tilde{X}_n = \big\{[x_1, \dots, x_m] \mid (x_1, \dots, x_m) \text{ pointed cube path in } X, x_m \in X_n\big\}$; $\tilde{i} = [i]$
- $\tilde{\delta}^0_k [x_1, \dots, x_m] = \big\{\sigma = (y_1, \dots, y_p) \mid y_p = \delta^0_k x_m, \sigma * x_m \sim (x_1, \dots, x_m)\big\}$
- $\tilde{\delta}^1_k [x_1, \dots, x_m] = [x_1, \dots, x_m, \delta^1_k x_m]$
- $\pi_X [x_1, \dots, x_m] = x_m$

Proposition 4.3 *The unfolding $(\tilde{X}, \pi_X)$ of a HDA X is well-defined, and $\tilde{X}$ is a higher-dimensional tree. If X itself is a higher-dimensional tree, then the projection $\pi_X : \tilde{X} \to X$ is an isomorphism.*

Lemma 4.4 *If X is a higher-dimensional automaton and $(\tilde{x}_1, \dots, \tilde{x}_m)$ is a pointed cube path in $\tilde{X}$, then $(\pi_X \tilde{x}_1, \dots, \pi_X \tilde{x}_j) \in \tilde{x}_j$ for all $j = 1, \dots, m$.*

Lemma 4.5 *For any HDA X there is a unique lift r in any commutative diagram as below, for morphisms $g : P \to Q \in \mathsf{HDP}$, $p : P \to \tilde{X}, q : Q \to X \in \mathsf{HDA}$:*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & \tilde{X} \arrow[d, "\pi_X"] \\ Q \arrow[r, "q"'] \arrow[ur, dotted, "r"] & X \end{tikzcd}$$

Corollary 4.6 *Projections are open, and any HDA is bisimilar to its unfolding.* $\square$

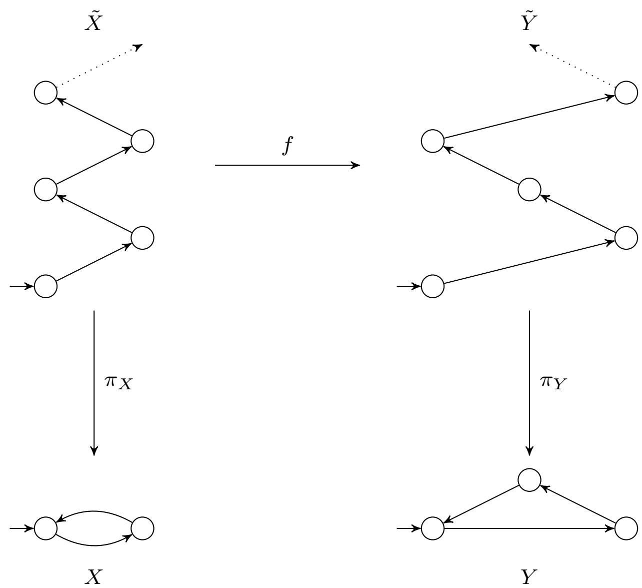

Fig. 5. Two simple one-dimensional HDA as objects of HDA and $\mathsf{HDA_h}$. In HDA there is no morphism $X \to Y$, in $\mathsf{HDA_h}$ there is precisely one morphism $f : X \to Y$.

5 History-Preserving Bisimilarity

In this section we recall history-preserving bisimilarity for HDA from [37] and show the main result of this paper: that hp-bisimilarity and the bisimilarity of Def. 3.2 are the same. To do this, we first need to introduce *morphisms of homotopy classes of paths* and *homotopy bisimilarity*.

Definition 5.1 The category of *higher-dimensional automata up to homotopy* $\mathsf{HDA_h}$ has as objects HDA and as morphisms pointed precubical morphisms $f : \tilde{X} \to \tilde{Y}$ of unfoldings.

Hence any morphism $X \to Y$ in HDA gives, by the unfolding functor, rise to a morphism $X \to Y$ in $\mathsf{HDA_h}$. The simple example in Fig. 5 shows that the converse is not the case. By restriction to higher-dimensional trees, we get a full subcategory $\mathsf{HDT_h} \hookrightarrow \mathsf{HDA_h}$.

Lemma 5.2 *The natural projection isomorphisms* $\pi_X : \tilde{X} \to X$ *for* $X \in \mathsf{HDT}$ *extend to an isomorphism of categories* $\mathsf{HDT_h} \cong \mathsf{HDT}$.

Restricting the above isomorphism to the subcategory HDP of HDT allows us to identify a subcategory $\mathsf{HDP_h}$ of $\mathsf{HDT_h}$ isomorphic to HDP.

Definition 5.3 A pointed morphism $f : X \to Y$ in $\mathsf{HDA_h}$ is *open* if it has the right lifting property with respect to $\mathsf{HDP_h}$, *i.e.* if it is the case that there is a lift r in any commutative diagram as below, for all morphism $g : P \to Q \in \mathsf{HDP_h}$,

$p : P \to X, q : Q \to Y \in \mathsf{HDA_h}$:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] \arrow[ur, dotted, "r"] & Y \end{tikzcd}$$

HDA X, Y are *homotopy bisimilar* if there is $Z \in \mathsf{HDA_h}$ and a span of open maps $X \leftarrow Z \to Y$ in $\mathsf{HDA_h}$.

The connections between open maps in $\mathsf{HDA_h}$ and open maps in HDA are as follows.

Lemma 5.4 *A morphism $f : X \to Y$ in $\mathsf{HDA_h}$ is open if and only if $f : \tilde{X} \to \tilde{Y}$ is open as a morphism of HDA. If $g : X \to Y$ is open in HDA, then so is $\tilde{g} : \tilde{X} \to \tilde{Y}$.*

We also need a lemma on prefixes in unfoldings.

Lemma 5.5 *Let X be a HDA and $\tilde{x}, \tilde{z} \in \tilde{X}$. Then there is a cube path from $\tilde{x}$ to $\tilde{z}$ in $\tilde{X}$ if and only if $\tilde{x} \sqsubseteq \tilde{z}$.*

Proposition 5.6 *For HDA $i : * \to X$, $j : * \to Y$, the following are equivalent:*

(i) *X and Y are homotopy bisimilar;*

(ii) *there exists a precubical subset $R \subseteq \tilde{X} \times \tilde{Y}$ with $(\tilde{i}, \tilde{j}) \in R$, and such that for all $(\tilde{x}_1, \tilde{y}_1) \in R$,*
- *for any $\tilde{x}_2 \in \tilde{X}$ for which $\tilde{x}_1 = \delta_k^0 \tilde{x}_2$ for some k, there exists $\tilde{y}_2 \in \tilde{Y}$ for which $\tilde{y}_1 = \delta_k^0 \tilde{y}_2$ and $(\tilde{x}_2, \tilde{y}_2) \in R$,*
- *for any $\tilde{y}_2 \in \tilde{Y}$ for which $\tilde{y}_1 = \delta_k^0 \tilde{y}_2$ for some k, there exists $\tilde{x}_2 \in \tilde{X}$ for which $\tilde{x}_1 = \delta_k^0 \tilde{x}_2$ and $(\tilde{x}_2, \tilde{y}_2) \in R$;*

(iii) *there exists a precubical subset $R \subseteq \tilde{X} \times \tilde{Y}$ with $(\tilde{i}, \tilde{j}) \in R$, and such that for all $(\tilde{x}_1, \tilde{y}_1) \in R$,*
- *for any cube path $(\tilde{x}_1, \dots, \tilde{x}_n)$ in $\tilde{X}$, there exists a cube path $(\tilde{y}_1, \dots, \tilde{y}_n)$ in $\tilde{Y}$ with $(\tilde{x}_p, \tilde{y}_p) \in R$ for all $p = 1, \dots, n$,*
- *for any cube path $(\tilde{y}_1, \dots, \tilde{y}_n)$ in $\tilde{Y}$, there exists a cube path $(\tilde{x}_1, \dots, \tilde{x}_n)$ in $\tilde{X}$ with $(\tilde{x}_p, \tilde{y}_p) \in R$ for all $p = 1, \dots, n$;*

(iv) *there exists a precubical subset $R \subseteq \tilde{X} \times \tilde{Y}$ with $(\tilde{i}, \tilde{j}) \in R$, and such that for all $(\tilde{x}_1, \tilde{y}_1) \in R$,*
- *for any $\tilde{x}_2 \sqsupseteq \tilde{x}_1$ in $\tilde{X}$, there exists $\tilde{y}_2 \sqsupseteq \tilde{y}_1$ in $\tilde{Y}$ for which $(\tilde{x}_2, \tilde{y}_2) \in R$,*
- *for any $\tilde{y}_2 \sqsupseteq \tilde{y}_1$ in $\tilde{Y}$, there exists $\tilde{x}_2 \sqsupseteq \tilde{x}_1$ in $\tilde{X}$ for which $(\tilde{x}_2, \tilde{y}_2) \in R$.*

Again, the requirement that R be a precubical subset is equivalent to saying that whenever $(\tilde{x}, \tilde{y}) \in R$, then also $(\delta_k^\nu \tilde{x}, \delta_k^\nu \tilde{y}) \in R$ for any k and $\nu \in \{0, 1\}$. The next result is what will allow us to relate hp-bisimilarity and bisimilarity.

Theorem 5.7 *HDA X, Y are homotopy bisimilar if and only if they are bisimilar.*

The following is an unlabeled version of hp-bisimilarity for HDA as defined in [37]:

Definition 5.8 HDA $i : * \to X$, $j : * \to Y$ are *history-preserving bisimilar* if there exists a relation R between pointed cube paths in X and pointed cube paths in Y for which $((i), (j)) \in R$, and such that for all $(\rho, \sigma) \in R$,

- for all $\rho' \sim \rho$, there exists $\sigma' \sim \sigma$ with $(\rho', \sigma') \in R$,
- for all $\sigma' \sim \sigma$, there exists $\rho' \sim \rho$ with $(\rho', \sigma') \in R$,
- for all $\rho' \sqsupseteq \rho$, there exists $\sigma' \sqsupseteq \sigma$ with $(\rho', \sigma') \in R$,
- for all $\sigma' \sqsupseteq \sigma$, there exists $\rho' \sqsupseteq \rho$ with $(\rho', \sigma') \in R$.

We are ready to show the main result of this paper, which together with Theorem 5.7 gives our characterization for hp-bisimilarity.

Theorem 5.9 *HDA X, Y are homotopy bisimilar if and only if they are history-preserving bisimilar.*

Corollary 5.10 *History-preserving bisimilarity is decidable for finite HDA.*

6 Labels

We finish this paper by showing how to introduce labels into the above framework of bisimilarity and homotopy bisimilarity. Also in the labeled case, we are able to show that the three notions of bisimilarity, homotopy bisimilarity and history-preserving bisimilarity agree.

For labeling HDA, we need a subcategory of pCub isomorphic to the category of sets and functions. Given a finite or countably infinite set $S = \{a_1, a_2, \dots\}$, we construct a precubical set $!S = \{!S_n\}$ by letting

$$!S_n = \big\{(a_{i_1}, \dots, a_{i_n}) \mid i_k \le i_{k+1} \text{ for all } k = 1, \dots, n-1\big\}$$

with face maps defined by $\delta_k^\nu(a_{i_1}, \dots, a_{i_n}) = (a_{i_1}, \dots, a_{i_{k-1}}, a_{i_{k+1}}, \dots, a_{i_n})$.

Definition 6.1 The category of *higher-dimensional tori* HDO is the full subcategory of pCub generated by the objects $!S$.

As any object in HDO has precisely one 0-cube, the pointed category $* \downarrow \mathsf{HDO}$ is isomorphic to HDO. It is not difficult to see that HDO is indeed isomorphic to the category of finite or countably infinite sets and functions, *cf.* [20].

Definition 6.2 The category of *labeled higher-dimensional automata* is the pointed arrow category $\mathsf{LHDA} = * \downarrow \mathsf{pCub} \to \mathsf{HDO}$, with objects $* \to X \to !S$ labeled pointed precubical sets and morphisms commutative diagrams

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] \arrow[d] & & Y \arrow[d] \\ !S \arrow[rr, "\sigma"'] & & !T \end{tikzcd}$$

Definition 6.3 A morphism $(f, \mathrm{id}) : (* \to X \to !S) \to (* \to Y \to !S)$ in LHDA is *open* if its component f is open in HDA. Labeled HDA $* \to X \to !S$, $* \to Y \to !S$ are *bisimilar* if there is $* \to Z \to !S \in \mathsf{LHDA}$ and a span of open maps $X \leftarrow Z \to Y$ in LHDA.

Next we establish a correspondence between split traces [37] and cube paths in higher-dimensional tori. For us, a *split trace* over a finite or countably infinite set S is a pointed cube path in $!S$. Hence *e.g.* a split trace $a^+b^+a^-b^+b^-$ (in the notation of [37]) corresponds to the cube path (i, a, ab, b, bb, b). Both indicate the start of an a event, followed by the start of a b event, the end of an a event, the start of a b event, and the end of a b event. Note that contrary to ST-traces [37], the split trace contains no information as to which of the two b events has terminated at the b^-.

By definition, a torus $!S$ on a finite or countably infinite set $S = \{a_1, a_2, \dots\}$ contains all n-cubes $(a_{i_1}, \dots, a_{i_n})$. Hence we have the following lemma:

Lemma 6.4 *Let $(x_1, \dots, x_m)$, $(y_1, \dots, y_m)$ be pointed cube paths in $!S$ with $x_m = y_m$. Then $(x_1, \dots, x_m) \sim (y_1, \dots, y_m)$.* □

Homotopy classes of split traces are thus determined by their endpoint and length:

Corollary 6.5 *The unfolding of a higher-dimensional torus $i : * \to !S \in \mathsf{HDO}$ is isomorphic to the pointed precubical set $j : * \to Y$ given as follows:*

- $Y_n = \{(x, m) \mid x \in !S_n, m \geq n, m \equiv n \mod 2\}$, $j = (i, 0)$
- $\delta_k^0(x, m) = (\delta_k^0 x, m-1)$, $\delta_k^1(x, m) = (\delta_k^1 x, m+1)$ □

The definitions of open maps and bisimilarity in $\mathsf{HDA_h}$ can now easily be extended to the labeled case. Again, we only need label-preserving morphisms.

Definition 6.6 The category of *labeled higher-dimensional automata up to homotopy* $\mathsf{LHDA_h}$ has as objects labeled HDA $* \to X \to !S$ and as morphisms pairs of precubical morphisms $(f, \sigma) : (* \to \tilde{X} \to !\tilde{S}) \to (* \to \tilde{Y} \to !\tilde{T})$ of unfoldings.

Definition 6.7 A morphism $(f, \mathrm{id}) : (* \to X \to !S) \to (* \to Y \to !S)$ in $\mathsf{LHDA_h}$ is *open* if its component f is open in $\mathsf{HDA_h}$. Labeled HDA $* \to X \to !S$, $* \to Y \to !S$ are *homotopy bisimilar* if there is $* \to Z \to !S \in \mathsf{LHDA_h}$ and a span of open maps $X \leftarrow Z \to Y$ in $\mathsf{LHDA_h}$.

The proof of the next theorem is exactly the same as the one for Theorem 5.7.

Theorem 6.8 *Labeled HDA X, Y are homotopy bisimilar if and only if they are bisimilar.* □

Also for the labeled version, we can now show that homotopy bisimilarity agrees with history-preserving bisimilarity. We first recall the definition from [37], where we extend the labeling morphisms to cube paths by $\lambda(x_1, \dots, x_m) = (\lambda x_1, \dots, \lambda x_m)$:

Definition 6.9 Labeled HDA $* \xrightarrow{i} X \xrightarrow{\lambda} !S$, $* \xrightarrow{j} Y \xrightarrow{\mu} !S$ are *history-preserving bisimilar* if there exists a relation R between pointed cube paths in X and pointed cube paths in Y for which $((i),(j)) \in R$, and such that for all $(\rho,\sigma) \in R$,

- $\lambda(\rho) = \mu(\sigma)$,
- for all $\rho' \sim \rho$, there exists $\sigma' \sim \sigma$ with $(\rho',\sigma') \in R$,
- for all $\sigma' \sim \sigma$, there exists $\rho' \sim \rho$ with $(\rho',\sigma') \in R$,
- for all $\rho' \sqsupseteq \rho$, there exists $\sigma' \sqsupseteq \sigma$ with $(\rho',\sigma') \in R$,
- for all $\sigma' \sqsupseteq \sigma$, there exists $\rho' \sqsupseteq \rho$ with $(\rho',\sigma') \in R$.

Theorem 6.10 *Labeled HDA X, Y are homotopy bisimilar if and only if they are history-preserving bisimilar.*

7 Conclusion

We have shown that hp-bisimilarity for HDA can be characterized by spans of open maps in the category of pointed precubical sets, or equivalently by a zig-zag relation between cubes in all dimensions. Aside from implying decidability of hp-bisimilarity for HDA, and together with the results of [37], this confirms that HDA is a natural formalism for concurrency: not only does it generalize the main models for concurrency which people have been working with, but it also is remarkably simple and natural.

One major question which remains is whether also *hereditary* hp-bisimilarity can fit into our framework. Because of its back-tracking nature, it seems that simple unfoldings of HDA are not the right tools to use; one should rather consider some form of back-unfoldings of forward-unfoldings. Given the undecidability result of [23], it seems doubtful, however, that any characterization as simple as the one we have for hp-bisimilarity can be obtained.

Another important question is how HDA relate to other models for concurrency which are not present in the spectrum presented in [37]. One major such formalism is the one of *history-dependent automata* which have been introduced by Montanari and Pistore in [27,28] and have recently attracted attention in model learning [1,2]. We conjecture that up to hp-bisimilarity, HDA are equivalent to history-dependent automata.

With regard to the geometric interpretation of HDA as directed topological spaces, there are two open questions related to the work laid out in the paper: In [7] we show that morphisms in HDA are open if and only if their geometric realizations lift pointed directed paths. This shows that there are some connections to weak factorization systems [3] here which should be explored; see [24] for a related approach.

In [8] we relate homotopy of cube paths to directed homotopy of directed paths in the geometric realization. Based on this, one should be able to prove that the geometric realization of the unfolding of a higher-dimensional automaton is the same as the universal directed covering [11] of its geometric realization and hence that morphisms in HDA_h are open if and only if their geometric realizations lift dihomotopy classes of pointed dipaths.

The precise relation of our HDA unfolding to the one for Petri nets [21, 29] and other models for concurrency should also be worked out. A starting point for this research could be the work on symmetric event structures and their relation to presheaf categories in [35].

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Appendix: Proofs

Proof of Lemma 3.3. For the implication (i) $\Longrightarrow$ (ii), let $p : P \to X$ be a pointed cube path with P represented by $(p_1, \dots, p_m)$ and $p(p_m) = x_1$. Let p_{m+1} be a cube of dimension one higher than p_m, set $p_m = \delta_k^0 p_{m+1}$, and let Q be the higher-dimensional path represented by $(p_1, \dots, p_m, p_{m+1})$. Let $g : P \to Q$ be the inclusion, and define $q : Q \to Y$ by $q(p_j) = f(p(p_j))$ for $j = 1, \dots, m$ and $q(p_{m+1}) = y_2$. We have a lift $r : Q \to X$ and can set $x_2 = r(p_{m+1})$.

The implication (ii) $\Longrightarrow$ (iii) can easily be shown by induction. The case $y_m = \delta_k^0 y_{m+1}$ follows directly from (ii), and the case $y_{m+1} = \delta_k^1 y_m$ is clear by $\delta_k^1 \circ f = f \circ \delta_k^1$.

To finish the proof, we show the implication (iii) $\Longrightarrow$ (i). Let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & Y \end{tikzcd}$$

be a commutative diagram, with P represented by $(p_1, \dots, p_m)$. Up to isomorphism we can assume that Q is represented by $(p_1, \dots, p_m, p_{m+1}, \dots, p_t)$ and that g is the inclusion. The cube $p(p_m)$ is reachable in X, and $(q(p_m), \dots, q(p_t))$ is a cube path in Y which starts in $q(p_m) = f(p(p_m))$. Hence we have a cube path $(x_m, \dots, x_t)$ in X with $x_m = p(p_m)$ and $q(p_j) = f(x_j)$ for all $j = m, \dots, t$, and we can define a lift $r : Q \to X$ by $r(p_j) = p(p_j)$ for $j = 1, \dots, m$ and $r(p_j) = x_j$ for $j = m+1, \dots, t$. $\square$

Proof of Theorem 3.4. For the implication (i) $\Longrightarrow$ (ii), let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a span of open maps and define $R = \{(x, y) \in X \times Y \mid \exists z \in Z : x = f(z), y = g(z)\}$. Then $(i, j) \in R$ because f and g are pointed morphisms, and the other properties follow by Lemma 3.3. The implication (ii) $\Longrightarrow$ (iii) can be shown by a simple induction, and for the implication (iii) $\Longrightarrow$ (i), the projections give a span $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$ and are open by Lemma 3.3. $\square$

Proof of Lemma 4.1 (*cf.* [9, Ex. 2.15]). We can represent a cube path $(x_1, \dots, x_n, x)$ as above by an element $(p_1, \dots, p_n)$ of the symmetric group S_n by setting $p_n = k_n$ and, working backwards, $p_j = (\{1, \dots, n\} \setminus \{p_{j+1}, \dots, p_n\})[k_j]$, denoting by this the k_j-largest element of the set in parentheses. This introduces a bijection between the set of cube paths from the lower left corner of x to x on the one hand, and elements of S_n on the other hand, and under this bijection adjacencies of cube paths are transpositions in S_n. These generate all of S_n, hence all such cube paths are homotopic. $\square$

Proof of Proposition 4.3. Before proving the proposition, we need an auxiliary notion of *fan-shaped* cube path together with a technical lemma. Say that a cube path $(x_1,\dots,x_m)$ in a precubical set X, with $x_m\in X_n$, is fan-shaped if

$$x_j\in\begin{cases}X_0 & \text{for } 1\le j\le m-n \text{ odd},\\ X_1 & \text{for } 1\le j\le m-n \text{ even},\\ X_{n+j-m} & \text{for } m-n<j\le m\,.\end{cases}$$

Hence a fan-shaped cube path is a one-dimensional path up to the point where it needs to build up to hit the possibly high-dimensional end cube x_m.

Lemma A.1 *Any pointed cube path in a higher-dimensional automaton $i:*\to X$ is homotopic to a fan-shaped one.*

Proof. Let us first introduce some notation: For any pointed cube path $(x_1,\dots,x_m)$, let $n_1,\dots,n_m\in\mathbb{N}$ be such that $x_j\in X_{n_j}$ (hence n_j is the *dimension* of x_j), and let $T(x_1,\dots,x_m)=n_1+\dots+n_m$. An easy induction shows that $j-n_j$ is odd for all j. Also, $T(x_1,\dots,x_m)\ge\frac12(n_m^2+m-1)$, with equality if and only if $(x_1,\dots,x_m)$ is fan-shaped.

Next we show that $n_1+\dots+n_m\equiv\frac12(n_m^2+m-1)\mod 2$. By oddity of $j-n_j$ we have $\sum_{j=1}^m n_j-\sum_{j=1}^m j\equiv m\mod 2$, and also $\frac12(n_m^2+m-1)-\sum_{j=1}^m j=\frac12(n_m^2-m^2-1)\equiv m\mod 2$, hence the claim follows.

We can now finish the proof by showing how to convert a cube path $(x_1,\dots,x_m)$ with $T(x_1,\dots,x_m)>\frac12(n_m^2+m-1)$ into an adjacent cube path $(x_1',\dots,x_m')$ which has $T(x_1',\dots,x_m')=T(x_1,\dots,x_m)-2$, essentially by replacing one of its cubes, called x_ℓ below, with another one of dimension $n_\ell-2$.

If $(x_1,\dots,x_m)$ is a cube path which is not fan-shaped, then there is an index $\ell\in\{3,\dots,m-1\}$ for which $n_\ell\ge 2$, $x_{\ell-1}=\delta^0_{k_2}x_\ell$ for some k_2, and $x_{\ell+1}=\delta^1_{k_3}x_\ell$ for some k_3. Assuming ℓ to be the *least* such index, we must also have $x_{\ell-2}=\delta^0_{k_1}x_{\ell-1}$ for some k_1.

Now if $k_2<k_3$, then $\delta^0_{k_2}x_{\ell+1}=\delta^0_{k_2}\delta^1_{k_3}x_\ell=\delta^1_{k_3-1}\delta^0_{k_2}x_\ell=\delta^1_{k_3-1}x_{\ell-1}$ by the precubical identity (1), hence we can let $(x_1',\dots,x_m')$ be the cube path with $x_j'=x_j$ for $j\ne\ell$ and $x_\ell'=\delta^0_{k_2}x_{\ell+1}$.

If $k_2>k_3$, then similarly $\delta^1_{k_3}x_{\ell-1}=\delta^1_{k_3}\delta^0_{k_2}x_\ell=\delta^0_{k_2-1}\delta^1_{k_3}x_\ell=\delta^0_{k_2-1}x_{\ell+1}$, and we can let $x_j'=x_j$ for $j\ne\ell$ and $x_\ell'=\delta^1_{k_3}x_{\ell-1}$.

For the remaining case $k_2=k_3$, we replace $x_{\ell-1}$ by another cube of equal dimension first: If $k_1<k_2$, then $x_{\ell-2}=\delta^0_{k_1}\delta^0_{k_2}x_\ell=\delta^0_{k_2-1}\delta^0_{k_1}x_\ell$, hence the cube path $(x_1'',\dots,x_m'')$ with $x_j''=x_j$ for $j\ne\ell-1$ and $x_{\ell-1}''=\delta^0_{k_1}x_\ell$ is adjacent to $(x_1,\dots,x_m)$, and $T(x_1'',\dots,x_m'')=T(x_1,\dots,x_m)$. For this new cube path, we have $x_{\ell-2}''=\delta^0_{k_2-1}x_{\ell-1}''$, $x_{\ell-1}''=\delta^0_{k_1}x_\ell''$, and $x_{\ell+1}''=\delta^1_{k_3}x_\ell''$, and as $k_1<k_3$, we can apply to the cube path $(x_1'',\dots,x_m'')$ the argument for the case $k_2<k_3$ above.

If $k_1\ge k_2$, then $x_{\ell-2}=\delta^0_{k_1}\delta^0_{k_2}x_\ell=\delta^0_{k_2}\delta^0_{k_1+1}x_\ell$ by another application of the precubical identity (1). Hence we can let $x_j''=x_j$ for $j\ne\ell-1$ and $x_{\ell-1}''=\delta^0_{k_1+1}x_\ell$. Then $x_{\ell-2}''=\delta^0_{k_2}x_{\ell-1}''$, $x_{\ell-1}''=\delta^0_{k_1+1}x_\ell''$, and $x_{\ell+1}''=\delta^1_{k_3}x_\ell''$, and as $k_1+1>k_3$, we can apply the argument for the case $k_2>k_3$ above. □

Now for the proof of Proposition 4.3, it is clear that the structure maps $\tilde\delta^1_k$ are well-defined. For showing that also the mappings $\tilde{\delta}^0_k$ are well-defined, we note first that $\tilde{\delta}^0_k[x_1,\dots,x_m]$ is independent of the representative chosen for $[x_1,\dots,x_m]$: If $(x'_1,\dots,x'_m)\sim(x_1,\dots,x_m)$, then $(y_1,\dots,y_p)\in\tilde{\delta}^0_k[x'_1,\dots,x'_m]$ if and only if $y_p=\delta^0_k x'_m=\delta^0_k x_m$ and $(y_1,\dots,y_p,x'_m)=(y_1,\dots,y_p,x_m)\sim(x'_1,\dots,x'_m)\sim(x_1,\dots,x_m)$, if and only if $(y_1,\dots,y_p)\in\tilde{\delta}^0_k[x_1,\dots,x_m]$.

We are left with showing that $\tilde{\delta}^0_k[x_1,\dots,x_m]$ is non-empty. By Lemma A.1 there is a fan-shaped cube path $(x'_1,\dots,x'_m)\in[x_1,\dots,x_m]$, and by Lemma 4.1 we can assume that $x'_{m-1}=\delta^0_k x'_m=\delta^0_k x_m$, hence $(x'_1,\dots,x'_{m-1})\in\tilde{\delta}^0_k[x_1,\dots,x_m]$.

We need to show the precubical identity $\tilde{\delta}^\nu_k\tilde{\delta}^\mu_\ell=\tilde{\delta}^\mu_{\ell-1}\tilde{\delta}^\nu_k$ for $k<\ell$ and $\nu,\mu\in\{0,1\}$. For $\nu=\mu=1$ this is clear, and for $\nu=\mu=0$ one sees that $(y_1,\dots,y_p)\in\tilde{\delta}^0_k\tilde{\delta}^0_\ell[x_1,\dots,x_m]$ if and only if $y_p=\delta^0_k\delta^0_\ell x_m=\delta^0_{\ell-1}\delta^0_k x_m$ and $(x_1,\dots,x_m)\sim(y_1,\dots,y_p,\delta^0_\ell x_m,x_m)\sim(y_1,\dots,y_p,\delta^0_k x_m,x_m)$, by adjacency.

The cases $\nu=1$, $\mu=0$ and $\nu=0$, $\mu=1$ are similar to each other, so we only show the former. Let $(x'_1,\dots,x'_m)\in[x_1,\dots,x_m]$ be a fan-shaped cube path with $x'_{m-1}=\delta^0_\ell x'_m$, *cf.* Lemma 4.1. Then $\tilde{\delta}^1_k\tilde{\delta}^0_\ell[x_1,\dots,x_m]=\tilde{\delta}^1_k[x'_1,\dots,x'_{m-1}]=[x'_1,\dots,x'_{m-1},\delta^1_k x'_{m-1}]$. Now $\delta^1_k x'_{m-1}=\delta^1_k\delta^0_\ell x'_m=\delta^0_{\ell-1}\delta^1_k x_m$, and by adjacency, $(x'_1,\dots,x'_{m-1},\delta^1_k x'_{m-1},\delta^1_k x'_m)\sim(x'_1,\dots,x'_{m-1},x'_m,\delta^1_k x'_m)$, so that we have $(x'_1,\dots,x'_{m-1},\delta^1_k x'_{m-1})\in\tilde{\delta}^0_{\ell-1}[x'_1,\dots,x'_m,\delta^1_k x'_m]=\tilde{\delta}^0_{\ell-1}\tilde{\delta}^1_k[x'_1,\dots,x'_m]$.

For showing that the projection $\pi_X:\tilde{X}\to X$ is a precubical morphism, we note first that $\pi_X\tilde{\delta}^1_k[x_1,\dots,x_m]=\pi_X[x_1,\dots,x_m,\delta^1_k x_m]=\delta^1_k x_m=\delta^1_k\pi_X[x_1,\dots,x_m]$ as required. For $\tilde{\delta}^0_k$, let again $(x'_1,\dots,x'_m)\in[x_1,\dots,x_m]$ be a fan-shaped cube path with $x'_{m-1}=\delta^0_k x'_m$. Then $\pi_X\tilde{\delta}^0_k[x_1,\dots,x_m]=\pi_X[x'_1,\dots,x'_{m-1}]=x'_{m-1}=\delta^0_k x'_m=\delta^0_k x_m=\delta^0_k\pi_X[x_1,\dots,x_m]$.

The proof that $*\to\tilde{X}$ is a higher-dimensional tree follows from Lemma 4.4: Let $(\tilde{x}_1,\dots,\tilde{x}_m)$, $(\tilde{y}_1,\dots,\tilde{y}_m)$ be pointed cube paths in $\tilde{X}$ with $\tilde{x}_m=\tilde{y}_m$, then we need to prove that $(\tilde{x}_1,\dots,\tilde{x}_m)\sim(\tilde{y}_1,\dots,\tilde{y}_m)$. Let $x_j=\pi_X\tilde{x}_j$, $y_j=\pi_X\tilde{y}_j$ for $j=1,\dots,m$ be the projections, then $(x_1,\dots,x_m)$, $(y_1,\dots,y_m)$ are pointed cube paths in X. By Lemma 4.4, $(x_1,\dots,x_j)\in\tilde{x}_j$ and $(y_1,\dots,y_j)\in\tilde{y}_j$ for all $j=1,\dots,m$.

By $\tilde{x}_m=\tilde{y}_m$, we know that $(x_1,\dots,x_m)\sim(y_1,\dots,y_m)$. Let $(x_1,\dots,x_m)=(z^1_1,\dots,z^1_m)\sim\cdots\sim(z^p_1,\dots,z^p_m)=(y_1,\dots,y_m)$ be a sequence of adjacencies, and let $\tilde{z}^\ell_j=[z^\ell_1,\dots,z^\ell_j]$. This defines pointed cube paths $(\tilde{z}^\ell_1,\dots,\tilde{z}^\ell_m)$ in $\tilde{X}$; we show that $(\tilde{x}_1,\dots,\tilde{x}_m)=(\tilde{z}^1_1,\dots,\tilde{z}^1_m)\sim\cdots\sim(\tilde{z}^p_1,\dots,\tilde{z}^p_m)=(\tilde{y}_1,\dots,\tilde{y}_m)$ is a sequence of adjacencies:

Let $\ell\in\{1,\dots,p-1\}$, and let $\alpha\in\{1,\dots,m-1\}$ be the index such that $z^\ell_\alpha\neq z^{\ell+1}_\alpha$ and $z^\ell_j=z^{\ell+1}_j$ for all $j\neq\alpha$. Then $(z^\ell_1,\dots,z^\ell_j)=(z^{\ell+1}_1,\dots,z^{\ell+1}_j)$ for $j<\alpha$ and $(z^\ell_1,\dots,z^\ell_j)\sim(z^{\ell+1}_1,\dots,z^{\ell+1}_j)$ for $j>\alpha$, hence there is an adjacency $(\tilde{z}^\ell_1,\dots,\tilde{z}^\ell_m)\sim(\tilde{z}^{\ell+1}_1,\dots,\tilde{z}^{\ell+1}_m)$.

For the last claim of the proposition, if X itself is a higher-dimensional tree, then an inverse to π_X is given by mapping $x\in X$ to the unique equivalence class $[x_1,\dots,x_m]\in\tilde{X}$ of any pointed cube path $(x_1,\dots,x_m)$ in X with $x_m=x$. $\square$

Proof of Lemma 4.4. Let $x_j=\pi_X\tilde{x}_j$, for $j=1,\dots,m$, then $(x_1,\dots,x_m)$ is a pointed cube path in X. We show the claim by induction: We have $\tilde{x}_1=\tilde{i}=[i]=$

$[x_1]$, so assume that $(x_1, \dots, x_j) \in \tilde{x}_j$ for some $j \in \{1, \dots, m-1\}$. If $\tilde{x}_{j+1} = \tilde{\delta}^1_k \tilde{x}_j$ for some k, then $x_{j+1} = \delta^1_k x_j$, and $(x_1, \dots, x_{j+1}) \in \tilde{x}_{j+1}$ by definition of $\tilde{\delta}^1_k$. Similarly, if $\tilde{x}_j = \tilde{\delta}^0_k \tilde{x}_{j+1}$ for some k, then $x_j = \delta^0_k x_{j+1}$, and $(x_1, \dots, x_{j+1}) \in \tilde{x}_{j+1}$ by definition of $\tilde{\delta}^0_k$. $\square$

Proof of Lemma 4.5. Let $(\tilde{x}_1, \dots, \tilde{x}_m)$ be a pointed cube path in $\tilde{X}$, and write $x_j = \pi_X \tilde{x}_j$ for $j = 1, \dots, m$. Let $(x_1, \dots, x_m, y_1, \dots, y_p)$ be an extension in X and define $\tilde{y}_j = [x_1, \dots, x_m, y_1, \dots, y_j]$ for $j = 1, \dots, p$. Then $(\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_p)$ is the required extension in $\tilde{X}$, which is unique as $\tilde{X}$ is a higher-dimensional tree. $\square$

Proof of Lemma 5.2. Using the projection isomorphisms, any morphism $f : X \to Y$ in $\mathsf{HDT_h}$ can be "pulled down" to a morphism $\pi_Y \circ f \circ \pi_X^{-1} : X \to Y$ of HDT. $\square$

Proof of Lemma 5.4. For the forward implication of the first claim, let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & \tilde{X} \arrow[d, "f"] \\ Q \arrow[r, "q"'] & \tilde{Y} \end{tikzcd} \tag{2}$$

be a diagram in HDA with $g : P \to Q \in \mathsf{HDP}$; we need to find a lift $Q \to \tilde{X}$.

Using the isomorphisms $\pi_P : \tilde{P} \to P$, $\pi_Q : \tilde{Q} \to Q$, we can extend this diagram to the left; note that $\tilde{g} : \tilde{P} \to \tilde{Q}$ is a morphism of HDP:

$$\begin{tikzcd} \tilde{P} \arrow[r, "\cong"'] \arrow[rr, bend left, "p'"] \arrow[d, "\tilde{g}"'] & P \arrow[r, "p"'] \arrow[d, "g"'] & \tilde{X} \arrow[d, "f"] \\ \tilde{Q} \arrow[r, "\cong"] \arrow[rr, bend right, "q'"'] & Q \arrow[r, "q"] & \tilde{Y} \end{tikzcd} \tag{3}$$

Hence we have a diagram

$$\begin{tikzcd} P \arrow[r, "p'"] \arrow[d, "\tilde{g}"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q'"'] & Y \end{tikzcd}$$

in $\mathsf{HDA_h}$, and as $\tilde{g} : P \to Q$ is a morphism of $\mathsf{HDP_h}$, we have a lift $r : Q \to X$ in $\mathsf{HDA_h}$. This gives a morphism $r : \tilde{Q} \to \tilde{X} \in \mathsf{HDA}$ in Diagram (3), and by composition with the inverse of the isomorphism $\pi_Q : \tilde{Q} \to Q$, a lift $r' : Q \to \tilde{X} \in \mathsf{HDA}$ in Diagram (2).

For the back implication in the first claim, assume $f : \tilde{X} \to \tilde{Y} \in \mathsf{HDA}$ open and let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & Y \end{tikzcd}$$

be a diagram in $\mathsf{HDA_h}$ with $g : P \to Q \in \mathsf{HDP_h}$; we need to find a lift $Q \to X$. Transferring this diagram to the category HDA, we have

$$\begin{tikzcd} \tilde{P} \arrow[r, "p"] \arrow[d, "g"'] & \tilde{X} \arrow[d, "f"] \\ \tilde{Q} \arrow[r, "q"'] & \tilde{Y} \end{tikzcd}$$

and as $g : \tilde{P} \to \tilde{Q}$ is a morphism of HDP, we get the required lift.

To prove the second claim, let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "h"'] & \tilde{X} \arrow[d, "\tilde{g}"] \\ Q \arrow[r, "q"'] & \tilde{Y} \end{tikzcd}$$

be a diagram in HDA with $h : P \to Q \in \mathsf{HDP}$. We can extend it using the projection morphisms:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "h"'] & \tilde{X} \arrow[r, "\pi_X"] \arrow[d, "\tilde{g}"] & X \arrow[d, "g"] \\ Q \arrow[r, "q"'] & \tilde{Y} \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Because g is open in HDA, we hence have a lift

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "h"'] & \tilde{X} \arrow[r, "\pi_X"] \arrow[d, "\tilde{g}"] & X \arrow[d, "g"] \\ Q \arrow[r, "q"'] \arrow[urr, dotted, "r"] & \tilde{Y} \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

and Lemma 4.5 then gives the required lift r' in the diagram

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & \tilde{X} \arrow[d, "\pi_X"] \\ Q \arrow[r, "r"'] \arrow[ur, dotted, "r'"] & X \end{tikzcd}$$

□

Proof of Lemma 5.5. For the forward implication, let $(\tilde{x}, \tilde{y}_1, \dots, \tilde{y}_p)$ be a cube path in $\tilde{X}$ with $\tilde{y}_p = \tilde{z}$, let $(x_1, \dots, x_m) \in \tilde{x}$, and write $y_j = \pi_X \tilde{y}_j$ for all j. By Lemma 4.4, $(x_1, \dots, x_m, y_1, \dots, y_p) \in \tilde{z}$.

For the other direction, let $(x_1, \dots, x_m, y_1, \dots, y_p) \in \tilde{z}$ such that $(x_1, \dots, x_m) \in \tilde{x}$, and define $\tilde{y}_j = [x_1, \dots, x_m, y_1, \dots, y_j]$ for all j. Then $(\tilde{x}, \tilde{y}_1, \dots, \tilde{y}_p)$ is the required cube path from $\tilde{x}$ to $\tilde{z}$ in $\tilde{X}$. □

Proof of Proposition 5.6. The implication (i) $\Longrightarrow$ (ii) follows directly from Theorem 3.4, and (iii) can be proven from (ii) by induction. (We can omit the reachability condition from items (ii) and (iii) because any cube in an unfolding is reachable.) Equivalence of (iii) and (iv) is immediate from Lemma 5.5.

For the implication (iii) $\implies$ (i), we can use Theorem 3.4 to get a span $\tilde{X} \xleftarrow{f} R \xrightarrow{g} \tilde{Y}$ of open maps in HDA. Connecting these with the projection $\pi_R : \tilde{R} \to R$ gives a span $\tilde{X} \xleftarrow{f\circ\pi_R} \tilde{R} \xrightarrow{g\circ\pi_R} \tilde{Y}$. By Corollary 4.6, the maps in the span are open in HDA, hence by Lemma 5.4, $X \xleftarrow{f\circ\pi_R} R \xrightarrow{g\circ\pi_R} Y$ is a span of open maps in $\mathsf{HDA_h}$. □

Proof of Theorem 5.7. A span of open maps $X \xleftarrow{f} Z \xrightarrow{g} Y$ in HDA lifts to a span $X \xleftarrow{\tilde{f}} Z \xrightarrow{\tilde{g}} Y$ in $\mathsf{HDA_h}$, and $\tilde{f}$ and $\tilde{g}$ are open by Lemma 5.4. Hence bisimilarity implies homotopy bisimilarity.

For the other direction, let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a span of open maps in $\mathsf{HDA_h}$. In HDA, this is a span $\tilde{X} \xleftarrow{f} \tilde{Z} \xrightarrow{g} \tilde{Y}$, and composing with the projections yields $X \xleftarrow{\pi_X\circ f} \tilde{Z} \xrightarrow{\pi_Y\circ g} Y$. By Lemma 5.4 and Corollary 4.6, both $\pi_x \circ f$ and $\pi_Y \circ g$ are open in HDA. □

Proof of Theorem 5.9. For the "if" part of the theorem, assume that we have a relation R as in Definition 5.8 and define $\tilde{R} \subseteq \tilde{X}\times\tilde{Y}$ by $\tilde{R} = \{(\tilde{x}, \tilde{y}) \mid \exists \rho \in \tilde{x}, \sigma \in \tilde{y} : (\rho, \sigma) \in R\}$. Then $(\tilde{i}, \tilde{j}) \in \tilde{R}$. Now let $(\tilde{x}_1, \tilde{y}_1) \in \tilde{R}$ and $\tilde{x}_2 \sqsupseteq \tilde{x}_1$. We have $\rho_1 \in \tilde{x}_1$ and $\sigma_1 \in \tilde{y}_1$ for which $(\rho_1, \sigma_1) \in R$. Let $\rho_1' \in \tilde{x}_1$ and $\rho_2 \in \tilde{x}_2$ such that $\rho_2 \sqsupseteq \rho_1'$, then $\rho_1' \sim \rho_1$, hence we have $\sigma_1' \sim \sigma_1$ for which $(\rho_1', \sigma_1') \in R$. By $\rho_2 \sqsupseteq \rho_1'$ we also have $\sigma_2 \sqsupseteq \sigma_1'$ for which $(\rho_2, \sigma_2) \in R$, hence $(\tilde{x}_2 = [\rho_2], [\sigma_2]) \in \tilde{R}$ as was to be shown. The symmetric condition in Theorem 5.6(iv) can be shown analogously, and one easily sees that $\tilde{R}$ is indeed a precubical set.

For the other implication, let $\tilde{R} \subseteq \tilde{X}\times\tilde{Y}$ be a precubical set as in Theorem 5.6(iv) and define a relation of pointed cube paths by $R = \{(\rho, \sigma) \mid ([\rho], [\sigma]) \in \tilde{R}\}$. Then $((i), (j)) \in R$. Now let $(\rho, \sigma) \in R$, then also $(\rho', \sigma') \in R$ for any $\rho' \sim \rho$, $\sigma' \sim \sigma$, showing the first two conditions of Definition 5.8. For the third one, let $\rho' \sqsupseteq \rho$, then $[\rho'] \sqsupseteq [\rho]$, hence we have $\tilde{y}_2 \sqsupseteq [\sigma]$ for which $([\rho'], \tilde{y}_2) \in \tilde{R}$. By definition of R we have $(\rho', \sigma') \in R$ for any $\sigma' \in \tilde{y}_2$, and by $\tilde{y}_2 \sqsupseteq [\sigma]$, there is $\sigma' \in \tilde{y}_2$ for which $\sigma' \sqsupseteq \sigma$, showing the third condition. The fourth condition is proved analogously. □

Proof of Corollary 5.10. The condition in Thm. 3.4(ii) immediately gives rise to a fixed-point algorithm similar to the one used to decide standard bisimilarity, *cf.* [26]. □

Proof of Theorem 6.10. The proof is similar to the one of Theorem 5.9. For the "if" part, the condition $\lambda(\rho) = \mu(\sigma)$ ensures that the homotopy bisimilarity relation respects homotopy classes of split traces, and for the "only if" part, starting with a homotopy bisimilarity relation $\tilde{R} \subseteq \tilde{X} \times \tilde{Y}$, we have to define the history-preserving bisimilarity relation R by $R = \{(\rho, \sigma) \mid ([\rho], [\sigma]) \in \tilde{R}, \lambda(\rho) = \mu(\sigma)\}$ instead. □